



ELSEVIER

Topology and its Applications 92 (1999) 237–245

**TOPOLOGY
AND ITS
APPLICATIONS**

Strongly hopfian manifolds as codimension-2 fibrators

Yongkuk Kim¹

Department of Mathematics, The University of Tennessee at Knoxville, Knoxville, TN 37996-1300, USA

Received 30 June 1997; received in revised form 18 September 1997

Abstract

If a closed n -manifold N has a 2–1 covering, we consider the covering space \tilde{N} of N corresponding to H , where H is the intersection of all subgroups H_i of index 2 in $\pi_1(N)$, i.e., $H = \bigcap_{i \in I} H_i$ with $[\pi_1(N) : H_i] = 2$ for $i \in I$. We will show that if $\pi_1(N)$ is residually finite, $\chi(N) \neq 0$, and \tilde{N} is hopfian, then N is a codimension-2 fibrator. And then, we will also get several results about codimension-2 fibrators as its corollaries. © 1999 Elsevier Science B.V. All rights reserved.

Keywords: Approximate fibration; Codimension-2 fibrator; Hopfian manifold; Residually finite group; Degree one mod 2 map; Continuity sets

AMS classification: Primary 57N15; 55R65, Secondary 57N25; 54B15

1. Introduction

Daverman [5] introduced the following definition: A closed n -manifold N^n is a *codimension-2 fibrator* (respectively, a *codimension-2 orientable fibrator*) if, whenever $p: M \rightarrow B$ is a proper map from an arbitrary (respectively, orientable) $(n+2)$ -manifold M to a 2-manifold B such that each $p^{-1}(b)$ is shape equivalent to N , then $p: M \rightarrow B$ is an approximate fibration. Then we have the following natural question:

Main question. Which manifolds N are codimension-2 fibrators?

In [5], Daverman showed that all simply connected manifolds, closed surfaces with nonzero Euler characteristic, and real projective n -spaces ($n > 1$) are codimension-2 fibrators. And he asked whether every closed n -manifold with a finite fundamental

¹ E-mail: ykim@math.utk.edu.

group is a codimension-2 fibrator. The answer for the case of N having a nonzero Euler characteristic is *yes* [1]. But, in general, surprisingly, the answer turned out to be *no* [7].

A group Γ is said to be *hopfian* if every epimorphism $f: \Gamma \rightarrow \Gamma$ is necessarily an isomorphism. A group Γ is said to be *residually finite* if for any nontrivial element x of Γ there is a homomorphism f from Γ onto a finite group K such that $f(x) \neq 1_K$. It is well known that every finitely generated residually finite group is hopfian. Call a closed manifold N *hopfian* if it is orientable and every degree one map $N \rightarrow N$ is a homotopy equivalence. In [6], Daverman showed the following theorem:

Every hopfian n -manifold N with hopfian $\pi_1(N)$ and a nonzero Euler characteristic is a codimension-2 orientable fibrator.

Whether it is a codimension-2 fibrator is still an open question. But, Chinen [1] showed that the answer is *yes* if N has no 2–1 covering. So it is natural to ask the following:

Question. What if N has a 2–1 covering?

In this paper, we will get a partial answer of that question, in a sense. If a closed n -manifold N has a 2–1 covering, we consider the covering space \tilde{N} of N corresponding to H , where H is the intersection of all subgroups H_i of index 2 in $\pi_1(N)$, i.e., $H = \bigcap_{i \in I} H_i$ with $[\pi_1(N) : H_i] = 2$ for $i \in I$. Then we see that, by Hall's Theorem (for any finitely generated group G , the number of subgroups of G having any fixed finite index is finite), the index set I is finite, and \tilde{N} is an n -dimensional orientable manifold, which follows from the facts that a (finite) covering space of an n -dimensional orientable manifold is an n -dimensional orientable manifold and any nonorientable manifold has a 2–1 orientable covering. We will show that if $\pi_1(N)$ is residually finite, $\chi(N) \neq 0$, and \tilde{N} is hopfian, then N is a codimension-2 fibrator. And then, we will also get several results about codimension-2 fibrators as its corollaries.

2. Preliminaries

Throughout this paper, the symbols \sim , \approx , and \cong denote homotopy equivalence, homeomorphism, and isomorphism in that order. The symbol χ is used to denote Euler characteristic. All manifolds are understood to be finite dimensional, connected, metric, and boundaryless. Whenever the presence of boundary is tolerated, the object will be called a manifold with boundary.

Approximate fibrations were introduced by Coram and Duvall [2] as a generalization of Hurewicz fibrations and cell-like maps. A proper map $p: M \rightarrow B$ between locally compact ANRs is called an *approximate fibration* if it has the following approximate homotopy lifting property: Given an open cover ε of B , an arbitrary space X , and two maps $g: X \rightarrow M$ and $F: X \times I \rightarrow B$ such that $p \circ g = F_0$, there exists a map $G: X \times I \rightarrow M$ such that $G_0 = g$ and $p \circ G$ is ε -close to F . The latter means: for each $z \in X \times I$ there exists an $U_z \in \varepsilon$ such that $\{F(z), p \circ G(z)\} \subset U_z$. Much of the theory of Hurewicz fibrations carries over to the set of approximate fibrations. For example, if a

proper map $p: M \rightarrow B$ is an approximate fibration, there is a homotopy exact sequence between M , B and fibers of p as follows:

$$\cdots \rightarrow \pi_{i+1}(B) \rightarrow \pi_i(p^{-1}b) \rightarrow \pi_i(M) \rightarrow \pi_i(B) \rightarrow \cdots.$$

Furthermore, the set of approximate fibrations is a closed subset of the space of maps between two compact ANRs with the sup-norm metric, while the set of Hurewicz fibrations may not be closed [3].

Let N^n be a closed manifold. A proper map $p: M \rightarrow B$ is N^n -like if each fiber $p^{-1}(b)$ is shape-equivalent to N . For simplicity or familiarity, we shall assume that each fiber $p^{-1}(b)$ in an N^n -like map to be an ANR having the homotopy type of N^n . Let N and N' be closed n -manifolds and $f: N \rightarrow N'$ be a map. If both N and N' are orientable, then the *degree of f* is the nonnegative integer d such that the induced endomorphism of $f_*: H_n(N; Z) \cong Z \rightarrow H_n(N'; Z) \cong Z$ amounts to multiplication by d , up to sign. In general, the *degree mod 2 of f* is the nonnegative integer d such that the induced endomorphism of

$$f_*: H_n(N; Z_2) \cong Z_2 \rightarrow H_n(N'; Z_2) \cong Z_2$$

amounts to multiplication by d . Any degree one mod 2 map $f: N \rightarrow N$ with $\chi(N) \neq 0$ induces a π_1 -epimorphism $f_\#: \pi_1(N) \rightarrow \pi_1(N)$ (see [1, Lemma 3.4]).

Suppose that N is a closed n -manifold and a proper map $p: M \rightarrow B$ is N -like. Let G be the set of all fibers, i.e., $G = \{p^{-1}(b): b \in B\}$. Put $C = \{p(g) \in B: g \in G \text{ and there exist a neighborhood } U_g \text{ of } g \text{ in } M \text{ and a retraction } R_g: U_g \rightarrow g \text{ such that } R_g|_{g'}: g' \rightarrow g \text{ is a degree one map for all } g' \in G \text{ with } g' \in G \text{ in } U_g\}$, and $C' = \{p(g) \in B: g \in G \text{ and there exist a neighborhood } U_g \text{ of } g \text{ in } M \text{ and a retraction } R_g: U_g \rightarrow g \text{ such that } R_g|_{g'}: g' \rightarrow g \text{ is a degree one mod 2 map for all } g' \in G \text{ with } g' \in G \text{ in } U_g\}$. Call C the *continuity set of p* and C' the *mod 2 continuity set of p* . D. Coram and P. Duvall [4] showed that C and C' are dense, open subsets of B .

The following [4, Proposition 2.8] is very useful for investigating codimension-2 fibrators.

Proposition 2.1. *If G is a usc decomposition of an orientable $(n+2)$ -manifold M into closed, orientable n -manifolds, then the decomposition space $B = M/G$ is a 2-manifold and $D = B \setminus C$ is locally finite in B , where C represents the continuity set of the decomposition map $p: M \rightarrow B$; if either M or some elements of G are nonorientable, B is a 2-manifold with boundary (possibly empty) and $D' = (\text{int } B) \setminus C'$ is locally finite in B , where C' represents the mod 2 continuity set of p .*

And, the following ([6, Theorem 2.2] or [9]) gives us useful information connecting hopfian manifolds and hopfian fundamental groups.

Proposition 2.2. *A closed, orientable n -manifold N is a hopfian manifold if any one of the following conditions holds:*

- (1) $n \leq 4$ and $\pi_1(N)$ is hopfian;
- (2) $\pi_1(N)$ contains a nilpotent subgroup of finite index.

3. Strongly hopfian manifolds as codimension-2 fibrators

Definition. Let N be a closed n -manifold. N is *strongly hopfian* if \tilde{N} is hopfian, where (\tilde{N}, \tilde{q}) is the covering space of N corresponding to $H = \bigcap_{i \in I} H_i$ with $I = \{i: [\pi_1(N) : H_i] = 2\} \neq \emptyset$, and $\tilde{N} = N$ when $I = \emptyset$.

From now on, we reserve the symbols \tilde{N} and H for the above meanings.

Lemma 3.1. *Let N be a strongly hopfian closed n -manifold with residually finite fundamental group and nonzero Euler characteristic. If a proper map $p: M^{n+2} \rightarrow B^2$ from an $(n+2)$ -manifold M onto a 2-manifold B is N -like, then p is an approximate fibration over C' , where C' denotes the mod 2 continuity set of p .*

Proof. If $I = \emptyset$, then $N = \tilde{N}$ is hopfian. Since $\pi_1(N)$ is residually finite, it is hopfian. Hence N is a hopfian n -manifold with $\chi(N) \neq 0$ and hopfian fundamental group. By [6, Theorem 5.10], N is a codimension-2 orientable fibrator. Since $I = \emptyset$ implies that N has no 2–1 covering, by [1, Corollary 3.3], N is a codimension-2 fibrator. Now we assume that $I \neq \emptyset$. Set $G = \{p^{-1}(b): b \in B\}$. Fix $g_0 \in G$ with $p(g_0) \in C'$. Take a neighborhood U of $p(g_0)$ such that $p^{-1}(U)$ retracts to g_0 , and take a smaller connected neighborhood V of $p(g_0)$ such that $p^{-1}(V)$ deformation-retracts to g_0 in $p^{-1}(U)$. Call this retraction $R: p^{-1}(V) \rightarrow g_0$. Then, we have that $R_\#: \pi_1(p^{-1}(V)) \rightarrow \pi_1(g_0)$ is an epimorphism. By Coram and Duvall [3], it is enough to show that $p|_{p^{-1}(V)}: p^{-1}(V) \rightarrow V$ is an approximate fibration. Take the covering map $q: V^* \rightarrow p^{-1}(V)$ corresponding to $R_\#^{-1}(H)$. Since $[\pi_1(p^{-1}(V)) : R_\#^{-1}(H)] = [\pi_1(g_0) : H] < \infty$, q is finite. So, by [6, Lemma 2.5], it suffices to show that $p \circ q: V^* \rightarrow V$ is an approximate fibration.

Claim. *For all $g \in G$ with $g \subset p^{-1}(V)$, $q^{-1}(g) \equiv g^*$ and \tilde{N} have the same homotopy type.*

First, appealing to the method of the proof of claim in [1, Proposition 3.5], we see that for all $g \in G$ with $g \subset p^{-1}(V)$, $q^{-1}(g) \equiv g^*$ is connected. Hence $\pi_0(g^*) = 1$. Now, let $i: g \rightarrow p^{-1}(V)$, $i^*: g^* \rightarrow V^*$, $j: (p^{-1}(V)) \rightarrow (p^{-1}(V), g)$ and $j^*: V^* \rightarrow (V^*, g^*)$ be the inclusion maps for $g \in G$ with $g \subset p^{-1}(V)$. From the homotopy exact sequence of (V^*, g^*) and $(p^{-1}(V), g)$, we have the following diagram:

$$\begin{array}{ccccccccc}
 \pi_2(V^*, g^*) & \longrightarrow & \pi_1(g^*) & \xrightarrow{i_\#^*} & \pi_1(V^*) & \xrightarrow{j_\#^*} & \pi_1(V^*, g^*) & \longrightarrow & 1 \\
 \cong \downarrow & & (q|g^*)_\# \downarrow & & q_\# \downarrow & & \cong \downarrow & & \downarrow \\
 \pi_2(p^{-1}(V), g) & \longrightarrow & \pi_1(g) & \xrightarrow{i_\#} & \pi_1(p^{-1}(V)) & \xrightarrow{j_\#} & \pi_1(p^{-1}(V), g) & \longrightarrow & 1
 \end{array}$$

Since $R|_g: g \rightarrow g_0$ has degree one mod 2, by [1, Lemma 3.4], the induced map $(R|_g)_\#: \pi_1(g) \rightarrow \pi_1(g_0)$ is onto, so it is an isomorphism by the fact that $\pi_1(g) = \pi_1(g_0)$ is hopfian. Since $R|_g = R \circ i$, $i_\#$ is a monomorphism, and so is $i_\#^*$. We easily see that

$$K \equiv (q|g^*)_{\#}(\pi_1(g^*)) \subset H, \quad \text{for} \\ i_{\#}((R|g)_{\#}^{-1}(H)) \cong i_{\#}(H) \cong H.$$

Moreover, by the diagram chasing argument (using the serpent lemma (see [15, p. 141])), we have that $\pi_1(g)/K \cong \pi_1(p^{-1}(V))/q_{\#}(\pi_1(V^*))$. Since

$$[\pi_1(g) : K] = [\pi_1(g) : H][H : K] \quad \text{and} \quad q_{\#}(\pi_1(V^*)) = R_{\#}^{-1}(H),$$

we have $[H : K] = 1$, i.e., $K = H$. It follows from the uniqueness of lifting that g^* and \tilde{N} have the same homotopy type.

Since $\chi(g) \neq 0$ and q is finite, $\chi(g^*) \neq 0$. And since every subgroup of a residually finite group is residually finite, $q_{\#}(\pi_1(g^*)) \cong \pi_1(g^*)$ is residually finite, and so $\pi_1(g^*)$ is hopfian. Recall that V^* is orientable. It follows from [6, Theorem 5.10] that $p^* : V^* \rightarrow B^* = V^*/G^* = V$ is an approximate fibration, where $G^* = \{g^* : g \in G \text{ with } g \subset p^{-1}(V)\}$ is the usc decomposition of V^* . \square

Lemma 3.2. *Let N be a strongly hopfian closed n -manifold with hopfian $\pi_1(N)$ and $\chi(N) \neq 0$. If an N -like proper map $p : M^{n+2} \rightarrow B^2$ from an $(n+2)$ -manifold onto a 2-manifold with boundary is an approximate fibration over $\text{int } B$, then $\partial B = \emptyset$.*

Proof. Suppose not. Then there exist $a_0 \in \partial B$, a neighborhood U of a_0 in B , and a deformation retract $R : p^{-1}(U) \rightarrow p^{-1}(a_0)$ such that

- (1) $U \approx$ the upper half plane $\{(x, y) \in E^2 \mid y \geq 0\}$,
- (2) $A = (\partial B) \cap U$ is an open arc, and
- (3) for all $a \in A$, $R|_{p^{-1}(a)} : p^{-1}(a) \rightarrow p^{-1}(a_0)$ is a homotopy equivalence.

Take the covering map $q : M^* \rightarrow p^{-1}(U)$ corresponding to H . Then by another argument similar to the proof in the Lemma 3.1, we have that for all $a \in A$, $q^{-1}(p^{-1}(a))$ is connected and $q^{-1}(p^{-1}(a)) \sim q^{-1}(p^{-1}(a_0)) \sim \tilde{N}$. And, since p is an approximate fibration over $p^{-1}(\text{int } U)$, for all $b, b' \in \text{int } U$, $q^{-1}(p^{-1}(b))_C \sim q^{-1}(p^{-1}(b'))_C \sim$ (say) N^* , where $q^{-1}(p^{-1}(b))_C$ and $q^{-1}(p^{-1}(b'))_C$ are components of $q^{-1}(p^{-1}(b))$ and $q^{-1}(p^{-1}(b'))$, respectively. Hence, by the fact of M^* is orientable and [5, Proposition 2.9], we see that for all $b \in \text{int } U$ and for all $a \in A$, the components $q^{-1}(p^{-1}(b))_C$ of $q^{-1}(p^{-1}(b))$ and $q^{-1}(p^{-1}(a))$ are orientable. Therefore, if $G^* = \{q^{-1}(p^{-1}(b))_C, q^{-1}(p^{-1}(a)) \mid b \in \text{int } U, a \in A\}$ is the usc decomposition of M^* , then by Proposition 2.1, $B^* = M^*/G^*$ is a 2-manifold without boundary.

Let $p^* : M^* \rightarrow B^*$ be the decomposition map and C^* be its continuity set. Since $p^*(q^{-1}(p^{-1}(A)))$ is homeomorphic to an open arc and $B^* \setminus C^*$ is locally finite in B^* , there is a point $a^* \in p^*(q^{-1}(p^{-1}(A))) \cap C^*$. So we have a map $N^* \sim q^{-1}(p^{-1}(b))_C \rightarrow q^{-1}(p^{-1}(a)) \sim \tilde{N}$ with degree one, where $p^*(q^{-1}(p^{-1}(a))) = a^*$ and for some $b \in \text{int } U$. Hence we have $\beta_i(N^*) \geq \beta_i(\tilde{N})$ for each i .

For $g \in G$ with $p(g) = b$, let $i : g \rightarrow p^{-1}(U)$ be the inclusion map. Set

$$\bar{H} = i_{\#}^{-1}(q_{\#}(\pi_1(M^*)) \cap i_{\#}(\pi_1(g))) = i_{\#}^{-1}(H \cap i_{\#}(\pi_1(g))) \quad \text{and} \\ K = (q|g_C^*)_{\#}(\pi_1(g_C^*)),$$

where $g_C^* = q^{-1}(p^{-1}(b))_C$. Then we can easily see that $K \subset \overline{H}$, $H \subset \overline{H}$. But since g^* has two (or more) components, $H \neq \overline{H}$. By [16, Proposition 11.1], we have $K = \overline{H}$. Now we take the covering map $N_{\overline{H}} \rightarrow g$ corresponding to \overline{H} , and take the covering map $N_H \rightarrow g$ corresponding to H . And since $H \subset \overline{H}$ and $H \neq \overline{H}$, we have a $d-1$ covering map $\tilde{N} \sim N_H \rightarrow N_{\overline{H}} \approx N_K \sim N^*$ with $d \geq 2$, so we have for each i , $\beta_i(\tilde{N}) \geq \beta_i(N^*)$ and $\chi(\tilde{N}) = d\chi(N^*)$. As before $\chi(N^*) = \chi(\tilde{N}) = d\chi(N^*)$, which gives a contradiction $\chi(\tilde{N}) = 0$. \square

Theorem 3.3. *A strongly hopfian n -manifold N with residually finite fundamental group and nonzero Euler characteristic is a codimension-2 fibrator.*

Proof. We may assume that $I \neq \emptyset$, i.e., N has a 2–1 covering. Let a proper map $p: M^{n+2} \rightarrow B^2$ from an $(n+2)$ -manifold M^{n+2} onto a 2-manifold B with boundary be N -like, and $G = \{p^{-1}(b): b \in B\}$. By Proposition 2.1, Lemmas 3.1 and 3.2, it suffices to show that p is an approximate fibration over $\text{int } B$. Let $D' = (\text{int } B) \setminus C'$. If $D' = \emptyset$, by the Lemma 3.1, there is nothing to prove. So assume that $D' \neq \emptyset$. Let $b_0 \in D$. We localize the situation so that $\text{int } B$ is an open disk containing $b_0 = p(g_0)$ and p is an approximate fibration over $(\text{int } B) \setminus b_0$. Also we may assume that $R: p^{-1}(\text{int } B) \rightarrow g_0$ is a strong deformation retraction. Take a covering $q: M^* \rightarrow p^{-1}(\text{int } B)$ corresponding to $R_{\#}^{-1}(H) = H$. By an argument similar to the proof of claims in the Lemma 3.1, we see that $g_0^* \equiv q^{-1}(g_0)$ is connected and has the homotopy type of \tilde{N} . Since

$$p|_{p^{-1}((\text{int } B) \setminus g_0)}: p^{-1}((\text{int } B) \setminus g_0) \rightarrow (\text{int } B) \setminus b_0$$

is an approximate fibration, for any $g, g' (\neq g_0) \in G$ in $p^{-1}(\text{int } B)$, their components $q^{-1}(g)_C \equiv g_C^*$ and $q^{-1}(g')_C \equiv g_C'^*$ have the same homotopy type. And since M^* is orientable, by [6, Proposition 2.9], g_C^* and g_0^* are orientable.

Now we follow the method of the proof in [6, Theorem 5.10], then we have

$$(R^*|_{g_C^*})_*: H_1(g_C^*) \rightarrow H_1(g_0^*)$$

is an epimorphism, where R^* is a lifting of R . By [6, Lemma 5.2'], $R^*|_{g_C^*}$ has a positive degree. It follows from [15, p. 399] that $\beta_i(g_C^*) \geq \beta_i(g_0^*)$ for each i .

Now, for $g (\neq g_0) \in G$, let $i: g \rightarrow p^{-1}(\text{int } B)$ be the inclusion map. Set

$$\begin{aligned} \overline{H} &= i_{\#}^{-1}(q_{\#}(\pi_1(M^*)) \cap i_{\#}(\pi_1(g))) = i_{\#}^{-1}(H \cap i_{\#}(\pi_1(g))) \quad \text{and} \\ K &= (q|_{g_C^*})_{\#}(\pi_1(g_C^*)). \end{aligned}$$

Then we can easily see that $K \subset \overline{H}$ and $H \subset \overline{H}$. And by [16, Proposition 11.1], we have $\overline{H} \subset K$, i.e., $K = \overline{H}$.

Now, let us examine the induced map $i_{\#}$ case by case.

Case 1. $i_{\#}$ is an epimorphism. Then, since $\pi_1(g) = \pi_1(p^{-1}(\text{int } B))$ is hopfian, $i_{\#}$ is an isomorphism. So we have that for all $g \in G$ with $p(g) \in \text{int } B$, g^* is connected and has the homotopy type of \tilde{N} . By the same proof as the Lemma 3.1, p is an approximate fibration over $\text{int } B$.

Case 2. $i_{\#}$ is not onto.

Subcase 1. $H = \overline{H} = K$. Then for all $g \in G$ with $p(g) \in \text{int } B$, g^* is connected and has the homotopy type of \tilde{N} . By the same reason of the proof in the Lemma 3.1, p is an approximate fibration over $\text{int } B$.

Subcase 2. $H \subset \overline{H} = K$ but $H \neq \overline{H}$. We will show that this case cannot happen. Take the covering map $N_{\overline{H}} \rightarrow g$ corresponding to \overline{H} , and take the covering map $N_H \rightarrow g$ corresponding to H . Consider

$$\begin{array}{ccc} & N_{\overline{H}} & \\ & \downarrow & \\ N_H & \longrightarrow & g \end{array}.$$

Since $H \subset \overline{H}$ and $H \neq \overline{H}$, we have a $d-1$ covering map $N_H \rightarrow N_{\overline{H}}$ with $d \geq 2$. By the facts of $g_0^* \sim \tilde{N} \approx N_H$ and $N_{\overline{H}} \approx N_K \approx g_C^*$ with $d \geq 2$, we see that $\beta_i(g_0^*) \geq \beta_i(g_C^*)$ for each i (from [8, Corollary 1]) and $\chi(g_0^*) = d\chi(g_C^*)$ with $d \geq 2$. But since we already have that $\beta_i(g_C^*) \geq \beta_i(g_0^*)$ for each i , $\chi(g_C^*) = \chi(g_0^*) = d\chi(g_C^*)$ with $d \geq 2$, which gives the contradiction $\chi(\tilde{N}) = \chi(g_0^*) = 0$. \square

Note. A subgroup of a hopfian group may not be hopfian, while every subgroup of a residually finite group is residually finite (see [17]). Call a group Γ *hereditarily hopfian* if every subgroup of Γ is hopfian. The preceding argument actually gives the more general result stated below:

Let N be a strongly hopfian n -manifold with $\chi(N) \neq 0$. If $\pi_1(N)$ is hereditarily hopfian, then N is a codimension-2 fibrator.

Remark. In the theorem, we cannot omit the condition $\chi(N) \neq 0$ (see [7, Theorem 2.1]).

Corollary 3.1. *Let N^n be a closed n -manifold with $\chi(N) \neq 0$. Then N is a codimension-2 fibrator if any one of the following conditions holds:*

- (1) $\pi_1(N)$ is abelian;
- (2) $\pi_1(N)$ is residually finite and $\pi_i(N) = 0$ for $1 < i < n-1$;
- (3) $n = 4$ and $\pi_1(N)$ is residually finite;
- (4) [1] $\pi_1(N)$ is finite;
- (5) [5] $n = 2$.

Proof. (1) *Case 1.* N has no 2–1 covering. Then, N must be orientable. Since $\pi_1(N)$ is abelian, it is nilpotent. By Proposition 2.2, N is hopfian. We have that N is a hopfian manifold with hopfian fundamental group and nonzero Euler characteristic, so by [6, Theorem 5.10], N is a codimension-2 orientable fibrator. By Chinen [1, Corollary 3.3], N is a codimension-2 fibrator.

Case 2. N has a 2–1 covering. Since a finitely generated abelian group is residually finite, $\pi_1(N)$ is residually finite. And, since $\pi_1(\tilde{N}) \cong H$ is abelian, \tilde{N} is hopfian, and so N is a strongly hopfian manifold. Hence, N is a codimension-2 fibrator.

(2) Note that for $i \geq 2$, $0 \rightarrow \pi_i(\tilde{N}) \rightarrow \pi_i(N) \rightarrow 0$. Since $\pi_i(N) = 0$ for $1 < i < n-1$, $\pi_i(\tilde{N}) = 0$ for $1 < i < n-1$. By Swarup [18, Lemma 1.1], N is strongly hopfian.

(3) and (4) By Proposition 2.2, N is strongly hopfian.

(5) This follows from the facts that any closed surface has a residually finite fundamental group [11] and Proposition 2.2. \square

Now, let us consider the following question:

Question. Is any finite product of codimension-2 fibrators a codimension-2 fibrator?, i.e., if N_1, N_2, \dots, N_k are closed manifolds which are codimension-2 fibrators, is $N_1 \times N_2 \times \dots \times N_k$ a codimension-2 fibrator?

The answer is not yet settled. But the answer is *yes* for the case of each N_j ($j = 1, 2, \dots, k$) a closed orientable surface ([12] and [13]). Here, we have an affirmative answer without assuming orientability for any N_j as follows:

Corollary 3.2. *Any finite product of closed surfaces which are codimension-2 fibrators is a codimension-2 fibrator.*

Proof. Let N_1, N_2, \dots, N_k be closed surfaces which are codimension-2 fibrators, and $N \equiv N_1 \times N_2 \times \dots \times N_k$. First, note that for all $j = 1, \dots, k$, $\chi(N_j) \neq 0$, for the torus and Klein bottle are the only examples of noncodimension-2 fibrators (see [5]). So we have $\chi(N) \neq 0$. Moreover,

$$\pi_1(N) \cong \bigoplus_{j=1}^k \pi_1(N_j)$$

is residually finite. Hence it suffices to show that N is strongly hopfian. If N has no 2–1 covering, then N must be orientable, so that each N_j is orientable. In [12] and [13], Im took care of this case. Hence we consider the case that N has a 2–1 covering. Since N is of the form *products of RP^2 \times products of S^2 \times products of closed surfaces which are neither RP^2 nor S^2* , \tilde{N} must be of the form *products of S^2 \times products of S^2 \times products of closed surfaces which are aspherical*. Hence \tilde{N} is hopfian, which follows from the fact that any finite product of simply connected manifolds and aspherical closed manifolds with hopfian fundamental groups is hopfian (see [14]). \square

In closing, we mention the following unsettled topics.

Question 1. If N and N' are closed strongly hopfian manifolds with residually finite fundamental groups and nonzero Euler characteristics, then is $N \times N'$ a codimension-2 fibrator? Furthermore, is any finite product of such manifolds a codimension-2 fibrator?

Question 2. What conditions on a closed manifold are necessary for being a codimension-2 fibrator? What if $\chi(N) \neq 0$?

Acknowledgement

It is a pleasure for me to thank my thesis advisor, Robert J. Daverman, for his support and guidance during this project.

References

- [1] N. Chinen, Manifolds with nonzero Euler characteristic and codimension-2 fibrators, *Topology Appl.* 86 (1998) 151–167.
- [2] D.S. Coram and P.F. Duvall, Approximate fibration, *Rocky Mountain J. Math.* 7 (1977) 275–288.
- [3] D.S. Coram and P.F. Duvall, Approximate fibration and a movability condition for maps, *Pacific J. Math.* 72 (1977) 41–56.
- [4] D.S. Coram and P.F. Duvall, Mappings from S^3 to S^2 whose point inverses have the shape of a circle, *General Topology Appl.* 10 (1979) 239–246.
- [5] R.J. Daverman, Submanifold decompositions that induce approximate fibrations, *Topology Appl.* 33 (1989) 173–184.
- [6] R.J. Daverman, Hyperhopfian groups and approximate fibrations, *Compositio Math.* 86 (1993) 159–176.
- [7] R.J. Daverman, Codimension-2 fibrators with finite fundamental groups, Preprint.
- [8] B. Eckman, Covering and Betti numbers, *Bull. Amer. Math. Soc.* 55 (1949) 95–101.
- [9] J.C. Hausmann, Geometric hopfian and nonhopfian situation, *Lecture Notes in Pure Appl. Math.* (Marcel Decker, Inc., New York, 1987) 157–166.
- [10] J. Hempel, 3-manifolds, *Annals of Math. Studies* 86 (Princeton Univ. Press, Princeton, NJ, 1976).
- [11] J. Hempel, Residual finiteness of surface groups, *Proc. Amer. Math. Soc.* 32 (1972) 323.
- [12] Young Ho Im, Products of surfaces that induce approximate fibrations, *Houston J. Math.* 21 (2) (1995) 339–348.
- [13] Young Ho Im, Decompositions into codimension two submanifolds that induce approximate fibrations, *Topology Appl.* 56 (1) (1994) 1–11.
- [14] Young Ho Im, Mee Kwang Kang and Ki Mun Woo, Codimension-2 fibrators that is closed under finite product, Preprint.
- [15] J.R. Munkres, *Elements of Algebraic Topology* (Addison-Wesley, New York, 1984).
- [16] W.S. Massey, *Algebraic Topology: An Introduction* (Springer, New York, 1977).
- [17] J. Roitberg, Residually finite, hopfian and co-hopfian spaces, *Contemporary Mathematics* 37 (1985) 131–144.
- [18] G.A. Swarup, On a theorem of C.B. Thomas, *J. London Math. Soc.* 8 (1974) 13–21.